Fifth solution.

$$LHS = \sum_{cycl}\left\{\frac{x^3y^3}{3} + \frac{x^3y^3}{3} + \frac{x^3z^3}{3}\right\} \overbrace{\geq}^{AM-GM} \sum_{cycl} x^3y^2z = RHS$$

Uche Eliezer Okeke

Sixth solution. Since by AM-GM Inequality
$2y^3 + z^3 \geq 3\sqrt[3]{(y^3)^2 z^3} = 3y^2z$ then

$$\sum_{cyc} x^3y^3 = \frac{1}{3}\sum_{cyc}\left(2x^3y^3 + z^3x^3\right) =$$

$$= \frac{1}{3}\sum_{cyc} x^3\left(2y^3 + z^3\right) \geq \frac{1}{3}\sum_{cyc} 3x^3y^2z = \sum_{cyc} x^3y^2z = xyz\sum_{cyc} x^2y.$$

Arkady Alt

Seventh solution. AGM.

$$(x^3y^3 + x^3y^3 + y^3z^3)/3 \geq x^3y^2z$$

so we have

$$\sum_{\text{cyc}} x^3y^3 =$$

$$= \frac{(xy)^3 + (xy)^3 + (yz)^3}{3} + \frac{(yz)^3 + (yz)^3 + (zx)^3}{3} + \frac{(zx)^3 + (zx)^3 + (xy)^3}{3} \geq$$

$$\geq xyz(x^2y + y^2z + z^2x)$$

Paolo Perfetti

Eighth solution. Let $x, y, z > 0$ be real numbers.
Then, let's prove that :

$$\frac{x^2y^2}{z} + \frac{y^2z^2}{x} + \frac{z^2x^2}{y} \geq x^2y + y^2z + z^2x.$$

Without lost of generality, suppose that $x \geq y \geq z$.
Then : $x^2y^2 \geq z^2x^2 \geq y^2z^2$ and $\frac{1}{z} \geq \frac{1}{y} \geq \frac{1}{x}$.
Consquently, by rearrengement inequality, we have :